THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIX. APRIL, 1912. NO. 4.

# SOME USEFUL MATHEMATICAL BOOKS BEYOND ELEMENTARY CALCULUS.

By DR. G. A. MILLER, University of Illinois.

There is probably always a considerable number of young men who would like to extend their knowledge of mathematics but who are in doubt as to the most useful books. This applies especially to those who do not have easy access to good libraries and who have never enjoyed good educational advantages. The following suggestions are intended mainly for this class, but it is hoped that some others may be able to derive profit from reading them. The fact that there are so many good recent American mathematical books is especially encouraging, and one of our objects is to call attention to this important fact.

We first inquire, what are the most useful works along the line of algebra for a student who has had only an elementary course in this subject. Doubtless all would agree that such a student should secure Bôcher's *Introduction to Higher Algebra,* published by the Macmillan Company in 1907. He may find Bôcher's treatment of many subjects limited and too concise, and it is hoped that he will be led to look elsewhere for greater details as well as for algebraic subjects which he does not find in this work; but he will probably appreciate the methods and the subject matter of this work more and more as he advances.

In particular, he may desire to learn something about the fundamental properties of numbers. In this case he would probably find Reid's *Theory of Algebraic Numbers,* published by the Macmillan Company in 1910, very inspiring as well as very elementary. Cajori's *Theory of Equations,* published by the same company in 1904, would serve very well to secure a deeper knowledge of the fundamental subject of equations, and this knowledge could be extended along important lines by reading Dickson's *Theory of Algebraic Equations,* published by Wiley and Sons in 1903. Among tne classic algebras in foreign languages the following are especially worth having: Weber's *Lehrbuch der Algebra,* second edition, published at Braunschweig, Germany, by Vieweg und Sohn, 1898-1908, three volumes of 703, 855, and

733 pages, respectively; Serret's *Algèbra Supérieure*, sixth edition, published at Paris, France, by Gauthier-Villars, 1910, two volumes of 648 and 694 pages, respectively; Capelli's *Istituzioni di Analisi Algebrica*, fourth edition, published at Napoli, Italy, by Pellerano, 1909, one volume of 953 pages.

The student who wishes to extend his knowledge of analysis beyond a first course in elementary calculus will find an abundance of good books suitable to various grades of advancement. Wilson's *Advanced Calculus*, recently published by Ginn and Company, and Hedrick's translation of Goursat's *Course in Mathematical Analysis*, Volume I, published by the same company, would doubtless serve to guide the student wisely on entering this vast and interesting field of mathematics. The two little volumes on differential equations by Cohen, published by Heath and Company, can also be heartily recommended in view of the great importance of the field to which they provide an easy introduction.

The vast field of function theory may be wisely entered either by means of volume I of the *Lectures on the Theory of Functions of Real Variables* by Pierpont, published a few years ago by Ginn and Company, or by means of the second edition of Forsyth's *Theory of Functions of a Complex Variable*, published by the Cambridge, England, University Press in 1900. The two books by Harkness and Morley, bearing the titles *Introduction to the Theory of Analytic Functions*, and *Treatise on the Theory of Functions*, published by Macmillan and Company, are also very good books for the beginner in this field.

Among the classic works on analysis in foreign languages, Jordan's *Cours d' Analyse*, published by Gauthier-Villars, Paris, France, and Picard's *Traité d' Analyse*, published by the same firm, are very celebrated, but there is a large number of other works along this line which deserve high praise, and which are extensively used. Osgood's *Lehrbuch der Funktionentheorie*, written by an American but published in German by Teubner of Leipzig, in 1907, constitutes one of the best introductions to this theory, and furnishes the shortest routes to many important questions.

The student who desires wise guidance in pursuing geometry beyond an elementary course in analytic geometry may safely begin with the *Projective Geometry* by Veblen and Young, published by Ginn and Company in 1910. Like Bôcher's *Algebra* this work will probably be appreciated more and more as the student advances, and it would be desirable to seek elsewhere for more complete information on many of the separate subjects. The *Introduction to Projective Geometry* by Emch, published by Wiley and Sons in 1905, could render valuable services along this line. Students who wish to enter the important field of differential geometry could find wise guidance in the *Treatise on Differential Geometry* by Eisenhart, published lately by Ginn and Company; and those who wish to enter the more special but far reaching field of projective differential geometry will naturally begin with Wilczynski's *Projective Differential Geometry*, published by Teubner of Leipzig, Germany, in 1906.

Among the classic works on geometry in foreign languages *La Thèorie Gênerale des Surfaces* by Darboux, published by Gauthier-Villars, Paris, France, is one of the most noted. This consists of four large volumes, and was published from 1837 to 1896. The second edition of Bianchi's *Legioni di Geometria Differenziale*, 1902-03, in two volumes of 523 and 594 pages, respectively, published by Spoerri, Pisa, Italy, is also very well known.

The preceding remarks relate separately to the three great fields of pure mathematics,— algebra, analysis, and geometry. A great deal of the most important mathematical literature does not limit itself to any one of these fields. Moreover, the student of mathematics needs outlook and independence as soon as he has sufficient knowledge to use these. That is, he needs books on books as well as thoughts on thoughts and ideas on ideas. Fortunately, such literature is growing very rapidly, although the bulk of it is in foreign languages. One of the most useful aids along this line in English is the Subject Index, Volume I, Pure Mathematics, of the Royal Society of London *Catalogue of Scientific Papers*, 1800-1900, published by the Cambridge, England, University Press, in 1908. This volume of about 700 pages is a subject index of the mathematical articles which appeared in 700 different serials during the nineteenth century, and is said to contain 38,748 entries.

Although this Index gives only the places where articles relating to different subjects were published it is of great value as one can often learn much in reference to what is probably new and what has been done by others from the various subjects to which articles relate and from the extent of these articles. The student of mathematics should never lose sight of the fact that good books are his tools and that it is almost as necessary for him to have the proper books as for the mechanic to have the proper tools. Only those who employ the best available tools for the work in hand have reason to expect notable success.

The Index mentioned above has been continued since 1900 by the annual mathematical volume in the *International Catalogue of Scientific Literature,* published for the International Council by the Royal Society of London. Each of the seventeen separate annual volumes of this catalogue contains both a subject and an author index, but it does not furnish more information about the article than the title conveys. The only extensive mathematical work which gives critical reviews of articles appearing from year to year is the excellent German publication entitled, *Jahrbuch über die Fortschritte der Mathematik.* About sixty different mathemations coöperate in providing reviews for this publication, which has appeared practically each year since 1871, and constitutes a most valuable source of information. The number of different articles reviewed at present is about three thousand per year and about one-third of a page is devoted, on an average, to each review.

As these reviewers are specialists of high scientific standing they have produced by their combined efforts a monumental work, which has had a most salutary influence on the development of mathematics, as it encour-

ages publications of high order and discourages research publications of little or no scientific value, as the latter publications are exposed to unfavorable reviews in this well known annual. It would be very desirable to have at least one more such annual, since even men of high scientific standing are not always free from prejudice. On the whole, it must be said that the reviews in the *Fortschritte* have been excellent and have more often dealt too leniently than too severely with the articles under consideration.

The most magnificent direct coöperation of mathematicians to produce a great work has been called into existence for the sake of completing the great German and French mathematical encyclopedias which are now in the course of publication by the firms of Teubner of Leipzig, Germany, and Gauthier-Villars of Paris, France. According to a recent circular more than 160 different mathematicians have been working on the German edition and more than 100 on the French. The vastness of the work may be inferred from the fact that the published parts of the German edition would fill more than a score of volumes of four hundred pages each although a large part of the field has not yet been covered; and, judging from the parts of the French that have appeared, this edition will be at least twice as large as the German. For instance, the number of pages in the published parts of Volume I of both of these two editions are as follows; the first number applying to the German edition. Fundamental principles of arithmetic, 27, 62; combinatory analysis and determinants, 19, 70; irrational numbers and convergence of infinite processes with real numbers, 100, 196; ordinary and higher complex numbers, 37, 140; infinite algorithms with complex numbers, 8, 20; theory of sets, 24, 42; finite discrete groups, 19, 85.

The object of this great work of reference is to give as completely as possible the fully established mathematical results and to exhibit by means of careful references the historical development of mathematical methods since the beginning of the nineteenth century. The work is not restricted to the so-called pure mathematics, but it includes applications to mechanics, physics, astronomy, geodesy, and various other technical subjects, so as to exhibit *in toto* the position occupied by mathematics in the present state of our civilization.

While such a vast work will naturally appeal to those interested in higher mathematics more than to those whose main interests are confined to the more elementary subjects, yet the latter will find much that is within their easy comprehension, especially in the introductory parts of arithmetic, geometry, and algebraic analysis. Moreover, it is of considerable value to read things once in a while which are not within one's easy comprehension. Great thoughts can sometimes be enjoyed even by those who cannot comprehend them completely, and a superficial view of the vastness of the developed parts of mathematics is much better than total ignorance and frequently awakens an interest in a particular field which appears to have received relatively too little attention. It may also serve to call attention to the im-

portant problem of mathematical transportation, or the utilization of the developments of one field in other fields, and the bringing together of facts whose similarity becomes striking through proximity. All the better libraries should be induced to subscribe for at least one of these great encyclopedias since such great works are of permanent value. Descriptive circulars can be obtained gratis by addressing the publishers as well as from some importers.

Among the less extensive works giving an outline of the main fields of mathematics which have been developed, the second edition of Pascal's *Repertorium der hoeheren Mathematik* takes the foremost place. The first two volumes, each covering more thon 500 pages, and costing about two dollars and a half, appeared in 1910, and the remaining two volumes are expected to appear towards the end of the present year. This work is also published by B. G. Teubner of Leipzig, Germany. In fact, this firm is now publishing more works on advanced mathematics than any other firm in the world.

Another very useful general work, which is, however, less modern in spirit, is Hagen's *Synopsis der hoeheren Mathematik,* consisting of three quarto volumes of about 400 pages each. This work was prepared in America when Hagen was Director of the Observatory of Georgetown College, but it was published in Germany by Dames, of Berlin. It may be of interest to observe that while the number of people who use the English language is very much larger than the number of those who use German, a few American mathematical writers have considered it best to publish their works in German. With the increase of scientific interest in China and Japan, where English is used much more than German or French, and with the growth of scientific interest among English speaking nations, it would appear that the English scientific works should soon command the most extensive market, and that more enterprising publishers of English scientific literature should come into existence.

There is a large number of other good general works on mathematics but the few that have been mentioned are among the very best, and they contain references to a large number of others. If any one should desire a cheaper guide through the mathematical literature but one which is fairly reliable and fairly extensive, he might be pleased with Mueller's *Fuehrer durch die mathematische Literature,* published by Teubner in 1909, and costing about two dollars.

In a very general way one may say that the German language contains the most along the lines of general literature, including the history of mathematics, as well as along algebraic and number theory lines. The French is richest along the line of analysis and comprehensive treatises on extensive fields of mathematics. The Italians are now ahead along geometrical lines, while the English excel in mathematical physics. It is hoped that the rapid recent advances in good English literature on various fields of pure

mathematics will have a good effect in creating a greater demand for such literature. Two recent American publications which may reasonably be expected to exert a good influence along this line are J. W. Young's *Fundamental Concepts of Algebra and Geometry,* and J. W. A. Young's *Monographs on Modern Mathematics.* These were published by the Macmillan Company, and by Longmans, Green and Company, respectively.

## NOTE ON THE BINOMIAL SERIES.

By K. OGURA in Sendai, Japan.

Euler proved the binomial theorem

$$F(y)=(1+x)^y=\sum_{n=0}^{\infty}\frac{y(y-1)\dots(y-n+1)}{n!}x^n \qquad |x|<1,$$

provided that $y$ is a rational number. There are various proofs of the above formula when $y$ is irrational.

But it is hoped that the following proof will be found both simple and rigorous.

In the binomial series

$$F(y)=\sum_{n=0}^{\infty}\frac{y(y-1)\dots(y-n+1)}{n.}x^n \qquad |x|<1,$$

put

$$f_n(y)=\frac{y(y-1)\dots(y-n+1)}{1.2\dots n}.$$

Let us denote by $G$ a finite positive number however great, and let us take $y$ so that $-G<y<G$.

(i) If $y>-1$, we can choose the positive numbers $k$ and $n$ so that $y+1<k<n$. Then

$$0<\frac{y+1}{k}<1, \quad 0<\frac{y+1}{k+1}<1, \dots, 0<\frac{y+1}{n}<1, \dots$$

and

$$|f_n(y)|=\left|(1-\frac{y+1}{1})(1-\frac{y+1}{2})\dots(1-\frac{y+1}{k-1})\right|\times(1-\frac{y+1}{k})\dots(1-\frac{y+1}{n}).$$

But since